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1980 J. Phys. A: Math. Gen. 13 2623

(http://iopscience.iop.org/0305-4470/13/8/011)

J. Phys. A: Math. Gen. 13 (1980) 2623–2629. Printed in Great Britain

Formulae for $y_{lm}(r_1 \times r_2)$

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Received 11 December 1979

Abstract. Equivalent formulae for $y_{lm}(r_1 \times r_2)$ are derived from boson and hyperspherical calculus. As a by-product, apparently new sum rules for $3jm$ symbols are obtained. Two passages from E_4 to E_3 are also discussed.

1. Introduction

Composition formulae for $\mathbb{R}^3$ regular solid harmonics $y_{lm}(\boldsymbol{r}) = r^l Y_{lm}(\theta, \phi)$ turn out to be very useful in various fields of physics and chemistry. For example, the translation formula (Rose 1958) for $y_{lm}(\boldsymbol{r}_1 - \boldsymbol{r}_2)$ plays a key role in the N-body theory as applied to nuclear (Eisenberg and Greiner 1970) and molecular (Steinborn and Ruedenberg 1973) physics. Throughout the course of work on the application of the Schwinger–Bargmann generating function method to the derivation of closed formulae for the Talmi and Moshinsky–Smirnov coefficients (Hage Hassan 1979, 1980a,b), a development of $y_{lm}(\boldsymbol{r}_1 \times \boldsymbol{r}_2)$ in terms of $y_{l_1 m_1}(\boldsymbol{r}_1)$ and $y_{l_2 m_2}(\boldsymbol{r}_2)$ has been obtained incidentally. It is the object of this short paper to report this development, along with two independent (at least seemingly distant) derivations. The first one makes use of the Schwinger–Bargmann calculus relative to the two-particle harmonic oscillator while the second one follows from a $\mathbb{R}^3$ projection of a composition formula relative to $\mathbb{R}^4$ solid harmonics. We first proceed with the two derivations, then deduce sum rules for $3jm$ coefficients and finally comment on two different ways of passing from E_4 to E_3.

2. First derivation

We start from the spherical basis of the three-dimensional harmonic oscillator for one particle. Following Bargmann and Moshinsky (1960), we have the boson representation

$$|nlm\rangle = \frac{(-1)^n}{n!\,2^{n+l/2}} N_{nl}\, \boldsymbol{a}^{+2n} y_{lm}(\boldsymbol{a}^+)|0\rangle$$
$$N_{nl} = \left(\frac{2\pi^{3/2}\Gamma(n+1)}{\Gamma(n+l+\frac{3}{2})}\right)^{1/2}. \qquad (1)$$

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In equation (1), y_{lm} denotes a normalised solid spherical harmonic (cf Edmonds 1957) and $\boldsymbol{a}^+ = (a_x^+, a_y^+, a_z^+)$ is a triplet of creation operators. We now go to the corresponding basis for two particles and use the completeness property of such a basis. Then

$$y_{lm}(\boldsymbol{a}_1^+ \times \boldsymbol{a}_2^+)|00\rangle = \sum_{\substack{n_1l_1m_1\\ n_2l_2m_2}} |n_1l_1m_1\rangle|n_2l_2m_2\rangle\langle n_1l_1m_1n_2l_2m_2|y_{lm}(\boldsymbol{a}_1^+ \times \boldsymbol{a}_2^+)|00\rangle. \tag{2}$$

By using equation (1) four times, equation (2) becomes

$$y_{lm}(\boldsymbol{a}_1^+ \times \boldsymbol{a}_2^+)|00\rangle$$
$$= \sum_{\substack{n_1l_1m_1\\ n_2l_2m_2}} \left(\frac{N_{n_1l_1}N_{n_2l_2}}{n_1!\,n_2!\,2^{n_1+n_2+(l_1+l_2)/2}}\right)^2 S\begin{pmatrix} n_1l_1m_1 & l \\ n_2l_2m_2 & m\end{pmatrix}$$
$$\times \boldsymbol{a}_1^{+2n_1} y_{l_1m_1}(\boldsymbol{a}_1^+)\boldsymbol{a}_2^{+2n_2} y_{l_2m_2}(\boldsymbol{a}_2^+)|00\rangle \tag{3}$$

where the S symbol is defined through

$$S\begin{pmatrix} n_1l_1m_1 & l \\ n_2l_2m_2 & m\end{pmatrix} = \langle 00|\boldsymbol{a}_1^{2n_1}\bar{y}_{l_1m_1}(\boldsymbol{a}_1)\boldsymbol{a}_2^{2n_2}\bar{y}_{l_2m_2}(\boldsymbol{a}_2)y_{lm}(\boldsymbol{a}_1^+ \times \boldsymbol{a}_2^+)|00\rangle. \tag{4}$$

The evaluation of the right-hand side of equation (4) may be performed by looking at the generating function of the S symbol. The latter generating function can be obtained from the generating function of the spherical harmonics (equation (5) below) and the generating function of the spherical basis for the three-dimensional harmonic oscillator (equation (6) below). From Schwinger (1965), we have

$$\left(\frac{4\pi}{2l+1}\right)^{1/2} \sum_m \Phi_{lm}(\zeta)y_{lm}(\boldsymbol{a}^+) = \frac{(\bar{\boldsymbol{\alpha}}\,.\,\boldsymbol{a}^+)^l}{2^l l!} \tag{5}$$

where

$$\zeta = (\xi, \eta), \qquad \Phi_{lm}(\zeta) = \sigma(l, m)\xi^{l+m}\eta^{l-m}, \qquad \sigma(l, m) = [(l+m)!\,(l-m)!]^{-1/2},$$
$$\boldsymbol{\alpha} = (\alpha_x, \alpha_y, \alpha_z), \qquad \alpha_x = -\xi^2 + \eta^2, \qquad \alpha_y = -\mathrm{i}(\xi^2+\eta^2), \qquad \alpha_z = 2\xi\eta.$$

(As is usual, $(\bar{\boldsymbol{v}}_1\,.\,\boldsymbol{v}_2) = v_{1x}v_{2x} + v_{1y}v_{2y} + v_{1z}v_{2z}$ so that $\boldsymbol{\alpha}$ is a null-length vector built from $\zeta \in \mathbb{C}^2$.) From Hage Hassan (1980a), we take

$$G(t, \boldsymbol{\alpha}, \boldsymbol{a}^+)|0\rangle = \exp\left(-\frac{t\boldsymbol{a}^{+2}}{2} + \frac{(\bar{\boldsymbol{\alpha}}\,.\,\boldsymbol{a}^+)}{2\sqrt{2}}\right)|0\rangle. \tag{6}$$

By combining equations (5) and (6), it is an easy affair to get the generating function of the S symbol:

$$\langle 00|G(\bar{t}_1, \bar{\boldsymbol{\alpha}}_1, \boldsymbol{a}_1)G(\bar{t}_2, \bar{\boldsymbol{\alpha}}_2, \boldsymbol{a}_2)\exp(\bar{\boldsymbol{\alpha}}\,.\,(\boldsymbol{a}_1^+ \times \boldsymbol{a}_2^+))|00\rangle$$
$$= \sum_{\substack{n_1l_1m_1\\ n_2l_2m_2\\ lm}} (-1)^{n_1+n_2}(4\pi)^{3/2}[(2l_1+1)(2l_2+1)(2l+1)]^{-1/2}$$
$$\times 2^l\bar{t}_1^{n_1}\bar{t}_2^{n_2}(n_1!\,n_2!\,2^{n_1+n_2+(l_1+l_2)/2})^{-1}$$
$$\times \Phi_{l_1m_1}(\bar{\zeta}_1)\Phi_{l_2m_2}(\bar{\zeta}_2)\Phi_{lm}(\zeta)S\begin{pmatrix} n_1l_1m_1 & l \\ n_2l_2m_2 & m\end{pmatrix}. \tag{7}$$

With the aid of the Schwinger boson calculus (Schwinger 1965) or the Bargmann integration procedure (Bargmann 1962), the left-hand side of equation (7) is found to be equal to

$$\exp[\tfrac{1}{16}\bar{t}_2(\bar{\boldsymbol{\alpha}}\cdot\bar{\boldsymbol{\alpha}}_1)^2+\tfrac{1}{16}\bar{t}_1(\bar{\boldsymbol{\alpha}}\cdot\bar{\boldsymbol{\alpha}}_2)^2+\tfrac{1}{8}(\bar{\boldsymbol{\alpha}}\cdot(\bar{\boldsymbol{\alpha}}_1\times\bar{\boldsymbol{\alpha}}_2))] \tag{8}$$

where

$$(\bar{\boldsymbol{\alpha}}\cdot\bar{\boldsymbol{\alpha}}_i)=2(\zeta\bar{\zeta}_i)^2$$
$$(\bar{\boldsymbol{\alpha}}\cdot(\bar{\boldsymbol{\alpha}}_1\times\bar{\boldsymbol{\alpha}}_2))=-4\mathrm{i}(\zeta\bar{\zeta}_1)(\zeta\bar{\zeta}_2)[\bar{\zeta}_1\bar{\zeta}_2]$$
$$(\zeta_1\zeta_2)=\xi_1\xi_2+\eta_1\eta_2, \qquad [\zeta_1\zeta_2]=\xi_1\eta_2-\xi_2\eta_1.$$

In the development of the expression (8), we insert the generating function for the $3jm$ symbol (Schwinger 1965):

$$\sum_{m_1m_2m}(-1)^{-l_1+l_2-m}\begin{pmatrix} l_1 & l_2 & l\\ m_1 & m_2 & -m\end{pmatrix}\Phi_{l_1m_1}(\bar{\zeta}_1)\Phi_{l_2m_2}(\bar{\zeta}_2)\Phi_{lm}(\zeta)$$
$$=\frac{(\zeta\bar{\zeta}_1)^{l_1-l_2+l}(\zeta\bar{\zeta}_2)^{-l_1+l_2+l}[\bar{\zeta}_1\bar{\zeta}_2]^{l_1+l_2-l}}{[(l_1+l_2+l+1)!\,(l_1-l_2+l)!\,(-l_1+l_2+l)!\,(l_1+l_2-l)!]^{1/2}}.$$

Comparison of the thus modified development and the right-hand side of equation (7) yields

$$S\begin{pmatrix} n_1l_1m_1 & l\\ n_2l_2m_2 & m\end{pmatrix}=\delta(n_1,\tfrac{1}{2}(l-l_1))\delta(n_2,\tfrac{1}{2}(l-l_2))\frac{(-1)^{m-l/2}}{(4\pi)^{3/2}2^l}[(2l_1+1)(2l_2+1)(2l+1)]^{1/2}$$
$$\times\left(\frac{(l_1+l_2+l+1)!(-l_1+l_2+l)!(l_1-l_2+l)!}{(l_1+l_2-l)!}\right)^{1/2}\begin{pmatrix} l_1 & l_2 & l\\ m_1 & m_2 & -m\end{pmatrix}. \tag{9}$$

By introducing equation (9) into equation (3), we finally obtain

$$y_{lm}(\boldsymbol{r}_1\times\boldsymbol{r}_2)=(-1)^{m-l/2}(4\pi)^{-3/2}2^{-l}(2l+1)^{1/2}\sum_{\substack{n_1l_1m_1\\ n_2l_2m_2}}\delta(n_1,\tfrac{1}{2}(l-l_1))\delta(n_2,\tfrac{1}{2}(l-l_2))$$
$$\times[(2l_1+1)(2l_2+1)]^{1/2}\left(\frac{N_{n_1l_1}N_{n_2l_2}}{n_1!\,n_2!\,2^{n_1+n_2+(l_1+l_2)/2}}\right)^2$$
$$\times\left(\frac{(l_1+l_2+l+1)!\,(-l_1+l_2+l)!\,(l_1-l_2+l)!}{(l_1+l_2-l)!}\right)^{1/2}$$
$$\times\begin{pmatrix} l_1 & l_2 & l\\ m_1 & m_2 & -m\end{pmatrix}r_1^{2n_1}y_{l_1m_1}(\boldsymbol{r}_1)r_2^{2n_2}y_{l_2m_2}(\boldsymbol{r}_2). \tag{10}$$

Equation (10) can be developed as

$$y_{lm}(\boldsymbol{r}_1\times\boldsymbol{r}_2)=(-1)^{m-l/2}[4\pi(2l+1)]^{1/2}(\tfrac{1}{2}r_1r_2)^l\sum_{\substack{l_1m_1\\ l_2m_2}}2^{l_1+l_2}\frac{[(2l_1+1)(2l_2+1)]^{1/2}}{(l+l_1+1)!\,(l+l_2+1)!}$$
$$\times\frac{[\tfrac{1}{2}(l+l_1)]!\,[\tfrac{1}{2}(l+l_2)]!}{[\tfrac{1}{2}(l-l_1)]!\,[\tfrac{1}{2}(l-l_2)]!}\left(\frac{(l_1+l_2+l+1)!\,(-l_1+l_2+l)!\,(l_1-l_2+l)!}{(l_1+l_2-l)!}\right)^{1/2}$$
$$\times\begin{pmatrix} l_1 & l_2 & l\\ m_1 & m_2 & -m\end{pmatrix}Y_{l_1m_1}(\theta_1,\phi_1)Y_{l_2m_2}(\theta_2,\phi_2) \tag{11}$$

which in turn yields

$$y_{lm}(\boldsymbol{r}_1\times\boldsymbol{r}_2)=(-1)^m(\sigma(l,m))^{-1}[4\pi(2l+1)]^{1/2}(\tfrac{1}{2}r_1r_2)^l$$
$$\times\sum_{\substack{l_1m_1\\l_2m_2}}\delta(m_1+m_2,m)2^{l_1+l_2}[(2l_1+1)(2l_2+1)]^{1/2}$$
$$\times\left(\frac{(l-l_1)!\,(l-l_2)!}{(l+l_1+1)!\,(l+l_2+1)!}\right)^{1/2}\frac{[\frac{1}{2}(l+l_1)]!\,[\frac{1}{2}(l+l_2)]!}{[\frac{1}{2}(l-l_1)]!\,[\frac{1}{2}(l-l_2)]!}$$
$$\times\sum_{\mu}(-1)^{\mu}\sigma(\tfrac{1}{2}l,\mu+m_1)\sigma(\tfrac{1}{2}l,-\mu+m_2)\begin{pmatrix}\frac{1}{2}l & \frac{1}{2}l & l_1\\ -\mu & \mu+m_1 & -m_1\end{pmatrix}$$
$$\times\begin{pmatrix}\frac{1}{2}l & \frac{1}{2}l & l_2\\ \mu & -\mu+m_2 & -m_2\end{pmatrix}Y_{l_1m_1}(\theta_1,\phi_1)Y_{l_2m_2}(\theta_2,\phi_2). \tag{12}$$

3. Second derivation

Let $q=(q_0,\boldsymbol{r})$ be a vector of the four-dimensional real Euclidean space E_4: $q_0\in\mathbb{R}$ and $\boldsymbol{r}=(x,y,z)\in E_3$ stand for the 'scalar' and 'vectorial' parts of q respectively. Moreover, let us put $|q|=(q_0^2+r^2)^{1/2}$. We take the $\mathbb{R}^4$ solid harmonics $y_{nlm}(q)$ to be defined by

$$y_{nlm}(q)=2^{l+1}|q|^{n-l-1}l!\left(\frac{n(n-l-1)!}{2\pi(n+l)!}\right)^{1/2}C^{l+1}_{n-l-1}(q_0/|q|)y_{lm}(\boldsymbol{r}) \tag{13}$$

where C^{l+1}_{n-l-1} is a Gegenbauer polynomial in the notation of Magnus *et al* (1966). (In equations (1) and (13), we employ the usual indices: do not confuse the two kinds of n.) The particular relations

$$C_0^{\lambda}(x)=1 \qquad \text{for any } x,$$
$$C_k^{\lambda}(0)=\begin{cases}0 & \text{for } k \text{ odd,}\\ (-1)^{k/2}\dfrac{\Gamma(\frac{1}{2}k+\lambda)}{\Gamma(\lambda)\Gamma(\frac{1}{2}k+1)} & \text{for } k \text{ even,}\end{cases} \tag{14}$$

will prove useful in the following.

The space E_4 may be endowed with a quaternionic structure by representing q via the following matrix of $\mathbb{R}_+\times SU_2$ (Talman 1968):

$$\check{q}=\begin{pmatrix}q_0+iz & -y+ix\\ y+ix & q_0-iz\end{pmatrix}.$$

From such a representation, we readily verify that the product of the two quaternions $q_1=(q_{01},\boldsymbol{r}_1)$ and $q_2=(q_{02},\boldsymbol{r}_2)$ is given by

$$q_1q_2=(q_{01}q_{02}-\boldsymbol{r}_1\cdot\boldsymbol{r}_2,\ q_{01}\boldsymbol{r}_2+q_{02}\boldsymbol{r}_1+\boldsymbol{r}_1\times\boldsymbol{r}_2). \tag{15}$$

Let us now introduce the homogeneous harmonic polynomials

$$\mathscr{D}^j(q)_{\mu_1\mu_2}=|q|^{2j}D^j(\check{q}/|q|)_{\mu_1\mu_2} \qquad \check{q}/|q|\in SU_2 \tag{16}$$

which constitute an extension to $\mathbb{R}_+\times SU_2$ of the matrix elements $D^j(\check{q}/|q|)_{\mu_1\mu_2}$ of the unitary irreducible representations of SU_2 ($\approx S^3$). The latter polynomials satisfy, of course, the multiplication formula

$$\mathscr{D}^j(q_1q_2)_{\mu_1\mu_2}=\sum_{\mu}\mathscr{D}^j(q_1)_{\mu_1\mu}\mathscr{D}^j(q_2)_{\mu\mu_2}. \tag{17}$$

The relation connecting the $y_{nlm}(q)$'s of equation (13) with the $\mathscr{D}^j(q)_{\mu_1\mu_2}$'s of equation (16) reads (Gazeau 1978)

$$y_{nlm}(q)=\frac{\mathrm{i}^l}{\pi}\left(\frac{n}{2}\right)^{1/2}\sum_{\mu_1\mu_2}(-1)^{(n-1)/2-\mu_2}(2l+1)^{1/2}\times\begin{pmatrix}\frac{1}{2}(n-1) & \frac{1}{2}(n-1) & l\\ \mu_1 & -\mu_2 & m\end{pmatrix}\mathscr{D}^{(n-1)/2}(q)_{\mu_1\mu_2} \tag{18}$$

or conversely

$$\mathscr{D}^{(n-1)/2}(q)_{\mu_1\mu_2}=(-1)^{(n-1)/2-\mu_2}\pi(2/n)^{1/2}\sum_{lm}(-\mathrm{i})^l(2l+1)^{1/2}\times\begin{pmatrix}\frac{1}{2}(n-1) & \frac{1}{2}(n-1) & l\\ \mu_1 & -\mu_2 & m\end{pmatrix}y_{nlm}(q). \tag{19}$$

Note that the phase factors in equations (18) and (19) differ from the ones of Bander and Itzykson (1966) and Sharp (1968), but that they are coherent with our defining relation (13).

We are now in a position to get the development of $y_{nlm}(q_1q_2)$ as function of $y_{n_1l_1m_1}(q_1)$ and $y_{n_2l_2m_2}(q_2)$. It is enough to develop $y_{nlm}(q_1q_2)$ owing to equation (18) and to use equation (17) once and equation (19) twice. This leads to a composition formula for the $\mathbb{R}^4$ solid harmonics:

$$y_{nlm}(q_1q_2)=(-1)^{m+n-1}\mathrm{i}^l(2l+1)^{1/2}\pi(2/n)^{1/2}\sum_{\substack{l_1m_1\\ l_2m_2}}(-\mathrm{i})^{l_1+l_2}[(2l_1+1)(2l_2+1)]^{1/2}\times\begin{Bmatrix}l_1 & l_2 & l\\ \frac{1}{2}(n-1) & \frac{1}{2}(n-1) & \frac{1}{2}(n-1)\end{Bmatrix}\begin{pmatrix}l_1 & l_2 & l\\ m_1 & m_2 & -m\end{pmatrix}y_{nl_1m_1}(q_1)y_{nl_2m_2}(q_2). \tag{20}$$

The last step amounts to projecting equation (20) from E_4 to E_3. This may be achieved by specialising equation (20) to the case $Q_1=(0,\boldsymbol{r}_1)$, $Q_2=(0,\boldsymbol{r}_2)$ and $n=l+1$. In such a case, the use of equations (14) and (15) gives

$$y_{l+1lm}(Q_1Q_2)=2^{l+1}l!\left(\frac{l+1}{2\pi(2l+1)!}\right)^{1/2}y_{lm}(\boldsymbol{r}_1\times\boldsymbol{r}_2)$$

$$y_{l+1l_im_i}(Q_i)=\begin{cases}0 & \text{when } l-l_i \text{ is odd}\\ (-1)^{(l-l_i)/2}2^{l_i+1}\left(\dfrac{(l+1)(l-l_i)!}{2\pi(l+l_i+1)!}\right)^{1/2}\dfrac{[\frac{1}{2}(l+l_i)]!}{[\frac{1}{2}(l-l_i)]!}r_i^{l-l_i}y_{l_im_i}(\boldsymbol{r}_i) & \\ \qquad\text{when } l-l_i \text{ is even,} & \end{cases}$$

so that we get the relation

$$y_{lm}(\boldsymbol{r}_1\times\boldsymbol{r}_2)=(-1)^{m+l/2}(2^l l!)^{-1}[4\pi(2l+1)(2l+1)!]^{1/2}$$
$$\times\sum_{\substack{l_1m_1\\l_2m_2}}2^{l_1+l_2}[(2l_1+1)(2l_2+1)]^{1/2}\left(\frac{(l-l_1)!\,(l-l_2)!}{(l+l_1+1)!\,(l+l_2+1)!}\right)^{1/2}$$
$$\times\frac{[\frac{1}{2}(l+l_1)]!\,[\frac{1}{2}(l+l_2)]!}{[\frac{1}{2}(l-l_1)]!\,[\frac{1}{2}(l-l_2)]!}\begin{Bmatrix}l_1 & l_2 & l\\ \frac{1}{2}l & \frac{1}{2}l & \frac{1}{2}l\end{Bmatrix}$$
$$\times\begin{pmatrix}l_1 & l_2 & l\\ m_1 & m_2 & -m\end{pmatrix}r_1^{l-l_1}y_{l_1m_1}(\boldsymbol{r}_1)r_2^{l-l_2}y_{l_2m_2}(\boldsymbol{r}_2),\qquad(21)$$

which can be seen to be equivalent to the central relation (11).

4. Sum rules

The three equivalent relations (11), (12) and (21) are valid in the case $\boldsymbol{r}_1=\boldsymbol{r}_2$. In that special case, they may be rewritten in the form of sum rules. For instance, we have

$$\sum_{l_1l_2}(-1)^{(l_1+l_2)/2}2^{l_1+l_2}(2l_1+1)(2l_2+1)\frac{[(l_1/2)!\,(l_2/2)!]^2}{l_1!\,l_2!}$$
$$\times\begin{Bmatrix}l_1 & l_2 & l\\ \frac{1}{2}l & \frac{1}{2}l & \frac{1}{2}l\end{Bmatrix}\begin{pmatrix}l_1 & l_2 & l\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix}l_1 & \frac{1}{2}l & \frac{1}{2}l\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix}l_2 & \frac{1}{2}l & \frac{1}{2}l\\ 0 & 0 & 0\end{pmatrix}=\delta(l,0),$$

or alternatively

$$\sum_{l_1l_2}2^{l_1+l_2}(2l_1+1)(2l_2+1)\left(\frac{(l-l_1)!\,(l-l_2)!}{(l+l_1+1)!\,(l+l_2+1)!}\right)^{1/2}$$
$$\times\frac{[\frac{1}{2}(l+l_1)]![\frac{1}{2}(l+l_2)]!}{[\frac{1}{2}(l-l_1)]![\frac{1}{2}(l-l_2)]!}\begin{Bmatrix}l_1 & l_2 & l\\ \frac{1}{2}l & \frac{1}{2}l & \frac{1}{2}l\end{Bmatrix}\begin{pmatrix}l_1 & l_2 & l\\ 0 & 0 & 0\end{pmatrix}=\delta(l,0)$$

$$\sum_{l_1l_2}2^{l_1+l_2}\frac{(2l_1+1)(2l_2+1)}{(l+l_1+1)!\,(l+l_2+1)!}\,\frac{[\frac{1}{2}(l+l_1)]![\frac{1}{2}(l+l_2)]!}{[\frac{1}{2}(l-l_1)]![\frac{1}{2}(l-l_2)]!}$$
$$\times\left(\frac{(l_1+l_2+l+1)!\,(-l_1+l_2+l)!\,(l_1-l_2+l)!}{(l_1+l_2-l)!}\right)^{1/2}\begin{pmatrix}l_1 & l_2 & l\\ 0 & 0 & 0\end{pmatrix}=\delta(l,0).$$

5. Concluding comment

To close this paper, it is perhaps worth noting that the two derivations rely on two distinct ways of passing from E_4 to E_3.

As already noticed, the second derivation calls for a simple projection

$$q=(q_0,\boldsymbol{r})\rightarrow Q=(0,\boldsymbol{r})$$

employed in conjunction with a little known relation connecting the polynomials

$\mathscr{D}^{l/2}(q)_{\mu_1\mu_2}$ and the harmonics $y_{lm}(\boldsymbol{r})$:

$$\sigma(l, m)y_{lm}(\boldsymbol{r}) = 2^{-l}\left(\frac{2l+1}{4\pi}\right)^{1/2} \sum_{\substack{\mu_1\mu_2 \\ -\mu_1+\mu_2=m}} (-1)^{\mu_1}$$

$$\times \sigma(\tfrac{1}{2}l, \mu_1)\sigma(\tfrac{1}{2}l, \mu_2)\mathscr{D}^{l/2}[(q_0, \boldsymbol{r})]_{\mu_1\mu_2} \qquad \forall q_0 \in \mathbb{R}$$

or

$$\mathscr{D}^{(l/2)}[(0, \boldsymbol{r})]_{\mu_1\mu_2} = (4\pi)^{1/2} \sum_{\substack{l'm' \\ l-l'=\text{even}}} (-1)^{l'+\mu_2} 2^{l'}(2l'+1)^{1/2}$$

$$\times \left(\frac{(l-l')!}{(l+l'+1)!}\right)^{1/2} \frac{[\frac{1}{2}(l+l')]!}{[\frac{1}{2}(l-l')]!} \begin{pmatrix} \frac{1}{2}l & \frac{1}{2}l & l' \\ \mu_1 & -\mu_2 & m' \end{pmatrix} r^{l-l'} y_{l'm'}(\boldsymbol{r}).$$

In contrast to the second derivation, the first one uses a well known relation (Talman 1968):

$$y_{lm}(\boldsymbol{r}) = \left(\frac{2l+1}{4\pi}\right)^{1/2} \overline{\mathscr{D}^l(Q)_{m0}}.$$

However, here the matrix

$$\check{Q} = \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix} \in \mathbb{R}_+ \times \mathrm{SU}_2$$

associated to the quaternion $Q \equiv Q(\boldsymbol{r})$ is deducible from $\boldsymbol{r}$ by means of nonlinear equations:

$$x = -(\alpha\beta + \bar{\alpha}\bar{\beta}) \qquad y = \mathrm{i}(\alpha\beta - \bar{\alpha}\bar{\beta}) \qquad z = |\alpha|^2 - |\beta|^2$$

with

$$\alpha\bar{\beta} + \bar{\alpha}\beta = 0.$$

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